Meeting 13 — Math 22

Reading Debrief

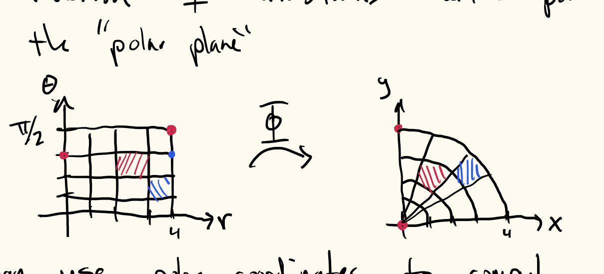
- Discuss Activity 11.5.2 w/ your group.
- Questions about polar coordinates?

Section 11.5.2 — Integrating in Polar Coordinates

The equations $x = r\cos\theta$ $y = r\sin\theta$ can be thought of as a function $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ where

$$\Phi(r,\theta) = (r\cos\theta, r\sin\theta).$$

The function Φ transforms the "Cartesian plane" into the "polar plane"



We can use polar coordinates to compute integrals $\iint_D e^{x^2+y^2}\,dA$ where D is the unit circle. The idea is to make the substitution $x = r\cos\theta$ $y = r\sin\theta$ everywhere.
However, the transformation Φ does not preserve area so we have to determine how to write dA in terms of r and θ.

Activity 11.5.3

- Complete together as a class.

Activity 11.5.3. Consider a polar rectangle R, with r between r_i and r_{i+1} and θ between θ_j and θ_{j+1} as shown at left in Figure 11.5.2. Let $\Delta r = r_{i+1} - r_i$ and $\Delta\theta = \theta_{j+1} - \theta_j$. Let ΔA be the area of this region.

a. Explain why the area ΔA in polar coordinates is not $\Delta r\,\Delta\theta$.

b. Now find ΔA by the following steps:

i. Find the area of the annulus (the washer-like region) between r_i and r_{i+1}, as shown at right in Figure 11.5.2. This area will be in terms of r_i and r_{i+1}.

ii. Observe that the region R is only a portion of the annulus, so the area ΔA of R is only a fraction of the area of the annulus. For instance, if $\theta_{i+1} - \theta_i$ were $\frac{\pi}{4}$, then the resulting wedge would be

$$\frac{\frac{\pi}{4}}{2\pi} = \frac{1}{8}$$

of the entire annulus. In this more general context, using the wedge between the two noted angles, what fraction of the area of the annulus is the area ΔA?

iii. Write an expression for ΔA in terms of r_i, r_{i+1}, θ_j, and θ_{j+1}.

iv. Finally, write the area ΔA in terms of r_i, r_{i+1}, Δr, and $\Delta\theta$, where each quantity appears only once in the expression. (Hint: Think about how to factor a difference of squares.)

c. As we take the limit as Δr and $\Delta\theta$ go to 0, Δr becomes dr, $\Delta\theta$ becomes $d\theta$, and ΔA becomes dA, the area element. Using your work in (iv), write dA in terms of r, dr, and $d\theta$.

b.i) $\pi r_{i+1}^2 - \pi r_i^2$
$= \pi(r_{i+1}^2 - r_i^2)$

b.ii) iii) $\pi(r_{i+1}^2 - r_i^2)\cdot\frac{\theta_{j+1}-\theta_j}{2\pi}$
$= \frac{(r_{i+1}^2 - r_i^2)(\theta_{j+1}-\theta_j)}{2} = \Delta A$

iv) $\frac{(r_{i+1}+r_i)\Delta r\,\Delta\theta}{2} = \Delta A$

c) $dA = \frac{2r\,dr\,d\theta}{2} = r\,dr\,d\theta$

Thus,

$$\boxed{\iint_D f(x,y)\,dA = \iint_D f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta.}$$

Example $f(x,y) = e^{x^2+y^2}$ D = unit disk $= \{(x,y) \mid x^2+y^2 \le 1\}$

$$\iint_D e^{x^2+y^2}\,dA = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} e^{x^2+y^2}\,dy\,dx$$

Can't do this integral ☹
Use polar coordinates:

$$\iint_D e^{x^2+y^2}\,dA = \iint_D e^{r^2}\,r\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^1 e^{r^2}\,r\,dr\,d\theta$$

$u = r^2$
$du = 2r\,dr$

$$= \frac{1}{2}\int_0^{2\pi}\int_0^1 e^u\,du\,d\theta$$
$$= \frac{1}{2}\int_0^{2\pi}\left[e^u\right]_0^1\,d\theta$$
$$= \frac{1}{2}\int_0^{2\pi} e-1\,d\theta = \pi(e-1).$$

Activity 11.5.4

- Complete w/ your group
- Class discussion.

Activity 11.5.4. Let $f(x,y) = x+y$ and $D = \{(x,y) : x^2+y^2 \le 4\}$.

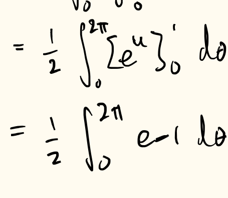
a. Sketch the region D and then write the double integral of f over D as an iterated integral in rectangular coordinates.

b. Write the double integral of f over D as an iterated integral in polar coordinates.

c. Evaluate one of the iterated integrals. Why is the final value you found not surprising?

a) Type I:

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} x+y\,dy\,dx$$



b) $$\int_0^{2\pi}\int_0^2 (r\cos\theta + r\sin\theta)\,r\,dr\,d\theta$$
$$= \int_0^{2\pi}(\cos\theta+\sin\theta)\left[\tfrac{1}{3}r^3\right]_0^2\,d\theta$$
$$= \frac{8}{3}\int_0^{2\pi}\cos\theta+\sin\theta\,d\theta = 0.$$

c) Clearly, the signed volume is zero.

Activity 11.5.5

- Complete w/ your group
- Class discussion.

Activity 11.5.5. Consider the circle given by $x^2+(y-1)^2 = 1$ as shown in Figure 11.5.4.

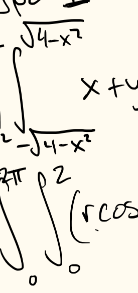


Figure 11.5.4. The graphs of $y = x$ and $x^2+(y-1)^2 = 1$, for use in Activity 11.5.5.

a. Determine a polar curve in the form $r = f(\theta)$ that traces out the circle $x^2+(y-1)^2 = 1$. (Hint: Recall that a circle centered at the origin of radius r can be described by the equations $x = r\cos(\theta)$ and $y = r\sin(\theta)$.)

b. Find the exact average value of $g(x,y) = \sqrt{x^2+y^2}$ over the interior of the circle $x^2+(y-1)^2 = 1$.

c. Find the volume under the surface $h(x,y) = x$ over the region D, where D is the region bounded above by the line $y = x$ and below by the circle (this is the shaded region in Figure 11.5.4).

d. Explain why in both (b) and (c) it is advantageous to use polar coordinates.

a) Replace x and y w/ $r\cos\theta$ and $r\sin\theta$ in the equation

$(r\cos\theta)^2 + (r\sin\theta - 1)^2 = 1$
$\Rightarrow r^2\cos^2\theta + r^2\sin^2\theta - 2r\sin\theta + 1 = 1$
$\Rightarrow r^2 - 2r\sin\theta = 0$
$\Rightarrow r(r - 2\sin\theta) = 0$
$\Rightarrow r = 2\sin\theta$

b) $$\iint_D g(x,y)\,dA = \iint_D g(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$
$$= \iint_D r^2\,dr\,d\theta$$
$$= \int_0^{\pi}\int_0^{2\sin\theta} r^2\,dr\,d\theta$$
$$= \frac{1}{3}\int_0^{\pi}\left[r^3\right]_0^{2\sin\theta}\,d\theta$$
$$= \frac{8}{3}\int_0^{\pi}\sin^3\theta\,d\theta$$

$u = \cos\theta$
$-du = \sin\theta\,d\theta$

$$= \frac{8}{3}\int_0^{\pi}\sin\theta(1-\cos^2\theta)\,d\theta$$
$$= -\frac{8}{3}\int_1^{-1} 1-u^2\,du = -\frac{8}{3}\left[u - \tfrac{1}{3}u^3\right]_1^{-1}$$
$$= -\frac{8}{3}\left[-1 - \tfrac{1}{3} - 1 + \tfrac{1}{3}\right]$$
$$= \frac{16}{3}$$

c) $$\iint_D h(x,y)\,dA = \iint_D r^2\cos\theta\,dr\,d\theta$$
$$= \int_0^{\pi/4}\int_0^{2\sin\theta} r^2\cos\theta\,dr\,d\theta$$
$$= \frac{1}{3}\int_0^{\pi/4}\left[r^3\cos\theta\right]_0^{2\sin\theta}\,d\theta$$
$$= \frac{8}{3}\int_0^{\pi/4}\sin^3\theta\cos\theta\,d\theta$$

$u = \sin\theta$
$du = \cos\theta\,d\theta$

$$= \frac{8}{3}\int_0^{\sqrt{2}/2} u^3\,du = \frac{8}{12}\left(\frac{\sqrt{2}}{2}\right)^4 = \frac{8\cdot 4}{12\cdot 16}$$
$$= \frac{1}{6}$$

Section 11.6

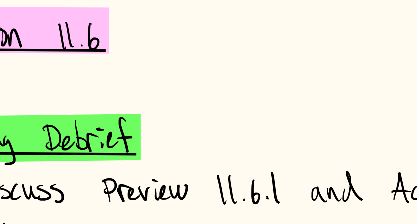
Reading Debrief

- Discuss Preview 11.6.1 and Activity 11.6.2 w/ your group.
- Questions?

Section 11.6.2 — Surface Area

Let $r(s,t) = \langle x(s,t), y(s,t), z(s,t)\rangle$ be a surface over a rectangle $a \le s \le b$ and $c \le t \le d$.

- Partition $[a,b]$ into m subintervals of equal length $\Delta s = \frac{b-a}{m}$ w/ $a = s_0 < s_1 < \cdots < s_m = b$
- Partition $[c,d]$ into n subintervals of equal length $\Delta t = \frac{d-c}{n}$ w/ $c = t_0 < t_1 < \cdots < t_n = d$.
- this gives us a partition of the rectangle $[a,b]\times[c,d]$ into $m\cdot n$ subrectangles of area $\Delta A = \Delta s\cdot\Delta t$



We need to approximate the area of the distorted blue parallelogram.
The function $r_s(s,t) = \langle x_s(s,t), y_s(s,t), z_s(s,t)\rangle$ gives a vector that is tangent to the surface in the s-direction. If we move a small distance Δs in the rectangle, then we move approx. $r_s\Delta s$ on the surface. Similarly, if we move Δt in the rectangle, we move approx $r_t\Delta t$ on the surface.
The vectors $r_s(s_{i-1},t_{j-1})\Delta s$ and $r_t(s_{i-1},t_{j-1})\Delta t$ span a parallelogram whose area is approximately the same as the distorted blue parallelogram. The area

$$|r_s(s_{i-1},t_{j-1})\Delta s \times r_t(s_{i-1},t_{j-1})\Delta t| = |r_s(s_{i-1},t_{j-1})\times r_t(s_{i-1},t_{j-1})|\Delta s\Delta t$$

To approximate the surface area, we sum them up

$$\sum_{j=1}^{n}\sum_{i=1}^{m} |r_s(s_{i-1},t_{j-1})\times r_t(s_{i-1},t_{j-1})|\Delta s\Delta t$$

Thus, the surface area is exactly

$$\boxed{\iint_R |r_s(s,t)\times r_t(s,t)|\,dA}$$

Activity 11.6.3

- Complete w/ your group. $\iint |r_s \times r_t|\,dA$
- Class discussion.

Activity 11.6.3. Consider the cylinder with radius a and height h defined parametrically by

$$\mathbf{r}(s,t) = a\cos(s)\mathbf{i} + a\sin(s)\mathbf{j} + t\mathbf{k}$$

for $0 \le s \le 2\pi$ and $0 \le t \le h$, as shown in Figure 11.6.7.

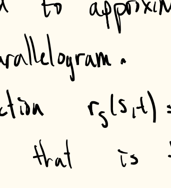


Figure 11.6.7. A cylinder.

a. Set up an iterated integral to determine the surface area of this cylinder.

b. Evaluate the iterated integral.

c. Recall that one way to think about the surface area of a cylinder is to cut the cylinder horizontally and find the perimeter of the resulting cross sectional circle, then multiply by the height. Calculate the surface area of the given cylinder using this alternate approach, and compare your work in (b).

a) $r_s = \langle -a\sin s, a\cos s, 0\rangle$
$r_t = \langle 0, 0, 1\rangle$

$$|r_s \times r_t| = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a\sin s & a\cos s & 0 \\ 0 & 0 & 1 \end{vmatrix}$$
$$= \begin{vmatrix} \hat{i} & \hat{j} \\ -a\sin s & a\cos s \end{vmatrix}$$
$$= |\langle a\cos s, a\sin s, 0\rangle|$$
$$= a.$$

(b)
$$\text{Surface area} = \iint_R a\,dA = \int_0^h\int_0^{2\pi} a\,ds\,dt = 2\pi ah.$$

Activity 11.6.4

- Complete w/ your group.